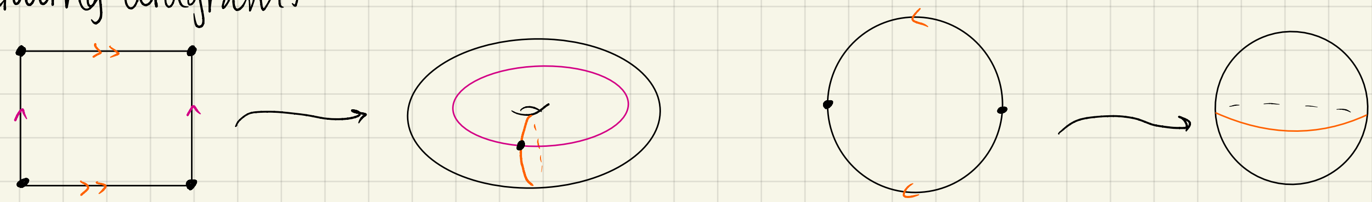
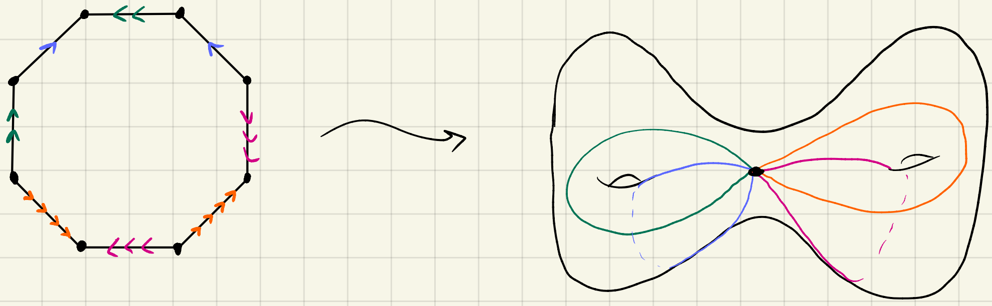
~ Introduction to topological spaces & homology

* CONSTRUCTION ZONE *

*we may construct topological spaces through a variety of methods.

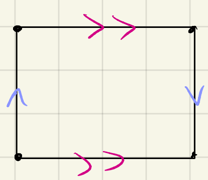
① Gluing diagrams





~ non-orientable surfaces:

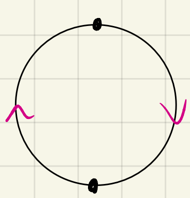
~ glue together 2 möbius bands to get a klein bottle



standard gluing diagram

~ in $\mathbb{R}^4$.
cannot be drawn in $\mathbb{R}^3$

~ $\mathbb{RP}^2$, the real projective plane



alternative defns: ① topological space of lines passing through the origin in $\mathbb{R}^3$.
② identifying antipodal points of the 2-sphere in $\mathbb{R}^3$.

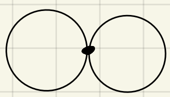
~ we can abstract this to $\mathbb{RP}^n$ ....

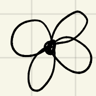
② making new spaces by building upon existing spaces.

* $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_f D^n$. this is called real projective space.

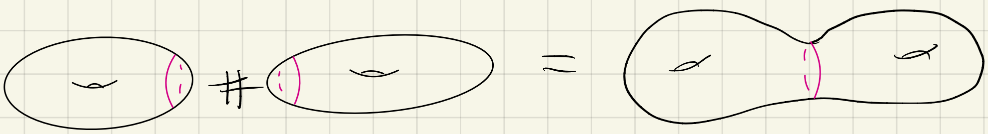
~ similarly, we may construct complex dimensional space.
$\mathbb{C}P^n$ is defined similarly, as lines passing through the origin of $\mathbb{C}^{n+1}$.
Notice here the dimensionality difference between $\mathbb{C}$ and $\mathbb{R}$.
the complex line contains objects of the form $a+bi$. Thus, we may relate $\mathbb{C}$ to $\mathbb{R}^2$. $\mathbb{C}^2$ to $\mathbb{R}^4$, etc.
~ the details of this construction is beyond the scope of this lecture.

~ wedge product

$S^1 \vee S^1$: 

$S^1 \vee S^1 \vee S^1 \vee S^1$:  bouquet of circles.

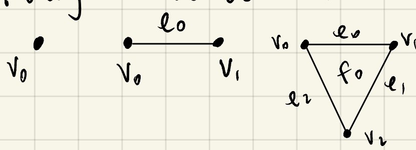
~ connect sum

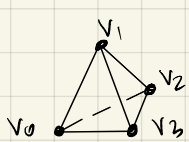


non-orientable: $T^2 \# \mathbb{R}P^2 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$

* Fact * this operation classifies all (closed, connected) surfaces. For those interested, it is called the classification theorem of closed surfaces.

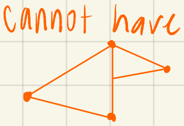
③ simplicial complex
~ they have a constructive defn.





tetrahedron

cannot have:  edge w/no vertices.

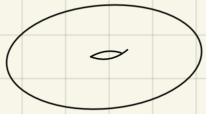
 intersection of Δs is not an edge. (or vertex)

✱ DISTINGUISHING SPACES ✱

Q: How do we tell the difference between 2 arbitrary topological spaces?

~ eg,

vs



?

~ the torus has a "hole."

✱ Essentially, homology tells us how many "holes" we have in each dimension. The input, for our sake, is a topological space and a group, usually $\mathbb{Z}$. The output is a group, in terms of $\mathbb{Z}$ (our input group), in each dimension.

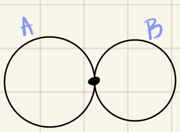
S^2:  → homology calculator →

dim 0 → $\mathbb{Z}$, for 1 connected component
dim 1 → 0, No 1-dimensional holes
dim 2 → $\mathbb{Z}$, one 2D hole.

B^3:  → homology calculator →

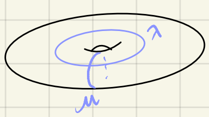
dim 0 → $\mathbb{Z}$, for 1 connected component
dim 1 → 0, No 1-dimensional holes
dim 2 → 0, no 2-dim hole.

this space has "no interesting homology"

$S^1 \vee S^1$: 

→ homology calculator →

dim 0 → $\mathbb{Z}$, for 1 connected component
dim 1 → $\mathbb{Z}_A \oplus \mathbb{Z}_B$, two circles, literally 1D holes

T^2: 

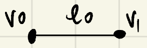
→ homology calculator →

dim 0 → $\mathbb{Z}$, for 1 connected component
dim 1 → $\mathbb{Z}_\lambda \oplus \mathbb{Z}_\mu$, two 1-D holes
dim 2 → $\mathbb{Z}$, one 2D hole.

✱ What we are really looking for are cycles that do not bound anything.

☆ Explicit computation ☆

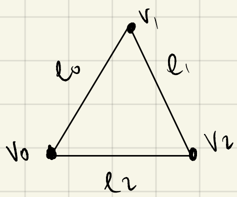
☆ We construct **chain groups** for each dimension that are related by **boundary maps** (homomorphisms) labeled ∂_n for dim = n.



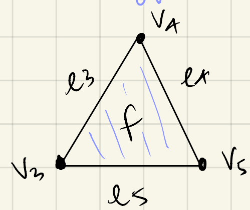
$\partial_1(e_0) = v_0 + v_1$ a non-interesting example. "literally" the boundary.

~ cycles vs boundaries note: this example uses $\mathbb{Z}_2$ coefficients. (Thus $\pm$ is irrelevant)

X =



y =



$\partial_1(e_0) = v_0 + v_1$
$\partial_1(e_1) = v_1 + v_2$
$\partial_1(e_2) = v_2 + v_0$

$\partial_1(e_3) = v_3 + v_4$
$\partial_1(e_4) = v_4 + v_5$
$\partial_1(e_5) = v_5 + v_3$

$\partial_2(f) = e_3 + e_4 + e_5$

~ we can represent these with matrices!
~ let's get slightly more technical.
F = faces, E = edges, V = vertices
then, $F \xrightarrow{\partial_2} E \xrightarrow{\partial_1} V \xrightarrow{\text{0-map}} 0$

☆ For space X, we have zero faces, so F = 0.

$$0 \xrightarrow{\partial_2 = 0} E \xrightarrow{\partial_1} V \xrightarrow{0} 0$$

only interesting map

$$\begin{array}{c|ccc} & e_0 & e_1 & e_2 \\ \hline v_0 & 1 & 1 & 0 \\ v_1 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 1 \end{array} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $e_0 + e_1 + e_2 \in \ker(\partial_1)$

☆ For y, 0 in dim = 3
$$0 \xrightarrow{0} F \xrightarrow{\partial_2} E \xrightarrow{\partial_1} V \xrightarrow{0} 0$$
$\partial_2(f) = e_3 + e_4 + e_5$, so $e_3 + e_4 + e_5 \in \operatorname{Im}(\partial_2)$

Also, $e_3 + e_4 + e_5 \in \ker(\partial_1)$.

☆ The V, E, & F are secretly the chain groups.

☆ Compute homology: $H_n(Z; \mathbb{Z}_2) = \ker\partial_n / \operatorname{Im}\partial_{n+1}$ where Z an arbitrary topological space.
also stated: cycles / boundaries.
again, looking for "holes."

~ we must get even more technical ~

$\operatorname{Im}\partial_2 = 0$!

$H_1(X; \mathbb{Z}_2) = \ker\partial_1 / \operatorname{Im}\partial_2 = \ker\partial_1 \rightsquigarrow$ homology sees our loop, $e_0 + e_1 + e_2$.

$H_1(y; \mathbb{Z}_2) = \ker\partial_1 / \operatorname{Im}\partial_2 = 0 \rightsquigarrow e_3 + e_4 + e_5$ is in both $\ker\partial_1$ and $\operatorname{Im}\partial_2$