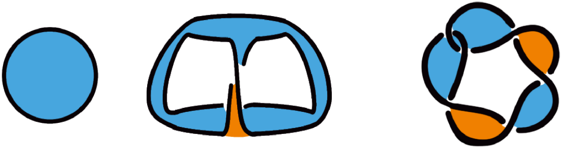
Non-Orientable Surfaces Bounded by Knots and the Knot Trace

Megan Fairchild

Louisiana State University

October 2024

Background - Slice Knots



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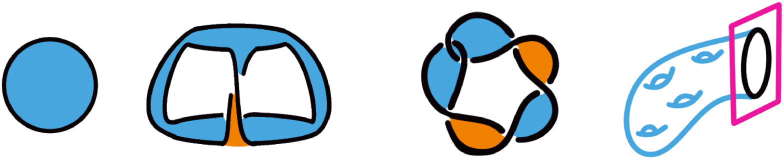

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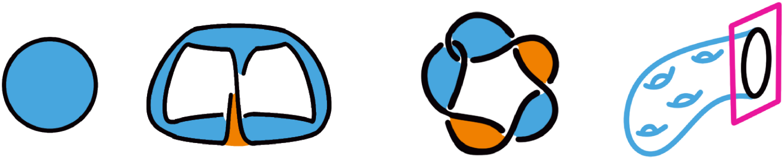

Definition (4-Genus)

Given a knot K in S^3, the 4-genus, $g_4(K)$, is defined to be the minimum genus among all orientable surfaces S smoothly embedded in B^4 so that $\partial S = K$.

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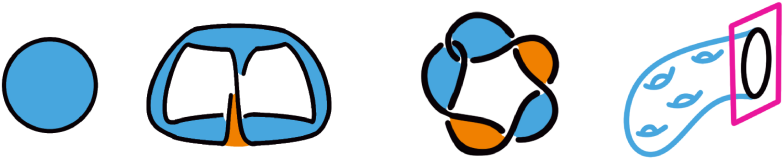

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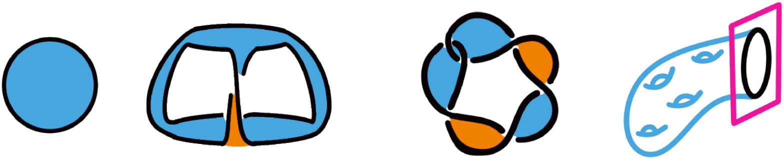

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- When $g_4(K) = 0$, we say K is a *slice knot*.

Background - Non-Orientable Analog

Definition (Non-Orientable 4 Genus)

Non-Orientable 4 genus is denoted $\gamma_4(K)$ and is defined to be the minimum first betti number of a surface F smoothly embedded in B^4 so that $\partial F = K$.

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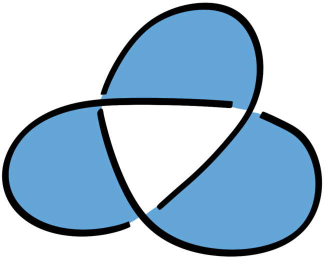

Motivation

- Does $g_4(K)$ provide a bound for $\gamma_4(K)$?

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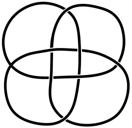
- What are the obstructions for $\gamma_4(K) = 1$?

$\gamma_4(K) = 2g_4(K) + 1$

- The 8_{18} knot has $g_4(8_{18}) = 1$ and $\gamma_4(8_{18}) = 3$



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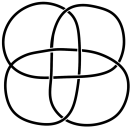

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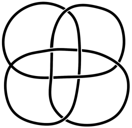

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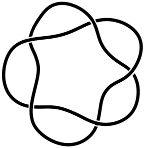

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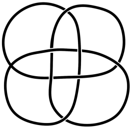

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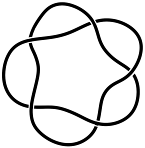

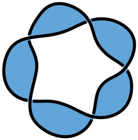

Thus, $\gamma_4(5_1) = 1$ and we have the bound $\gamma_4(K) \leq 2g_4(K) + 1$.

Techniques for Calculation

3 main methods for calculating the non-orientable 4-genus of knots.

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1. Lower bounds coming from knot invariants
2. Non-orientable band moves
3. Obstructions from the double branched cover

Knot Invariants

Denote the signature of a knot K as $\sigma(K)$ and the Arf invariant as $\mathrm{Arf}(K)$.

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Proposition (Yasuhara)

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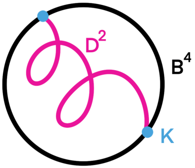

The 4-dimensional clasp number of a knot, $c_4(K)$, is the minimum number of double points of transversley immersed 2-disks in the 4-ball bounded by K.

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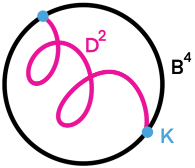

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Lemma (F)

Given a knot K satisfying $\sigma(K) + 4\mathrm{Arf}(\mathrm{K}) \equiv 4 \pmod 8$, and $c_4(K) \in \{1, 2\}$, then $\gamma_4(K) = 2$.

Knot Invariants - HFK

The little Upsilon invariant is denoted $\upsilon(K)$.

Proposition (Ozváth–Stipsicz–Szabó)

Given K is a knot,

$$\left| \upsilon(K) - \frac{\sigma(K)}{2} \right| \leq \gamma_4(K)$$

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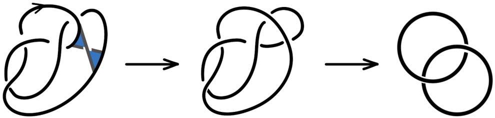
Theorem (Batson)

For a knot K,

$$\frac{\sigma(K)}{2} - d(S^3_{-1}(K)) \leq \gamma_4(K).$$

Band Moves & Cobordism

An orientable band move transforms a knot into a link.



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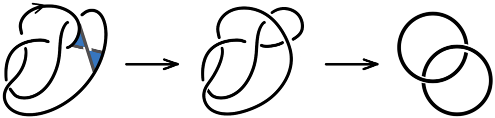


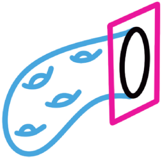
Figure 8 knot to Hopf Link

A non-orientable band move transforms a knot K into a different knot J.

Figure 8 knot to Trefoil

Transitioning to 4-Manifolds

Recall we have been discussing knots K in $S^3 = \partial B^4$ bounding surfaces in B^4.



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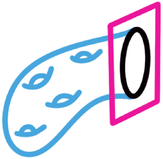

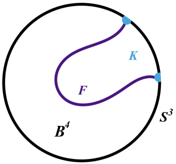

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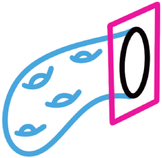

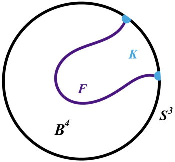

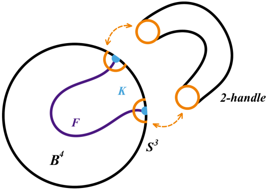

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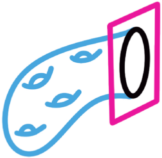
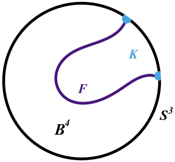
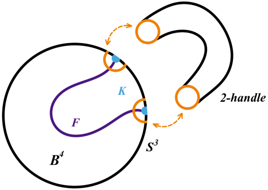


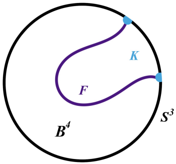




Question (Minimal Genus Problem)

What is the minimal genus of an embedded surface which represents a two-dimensional homology class in a closed oriented smooth 4-manifold?

Knot Trace



Let K be a knot in S^3 and a F a surface in B^4 with $\partial F = K$.

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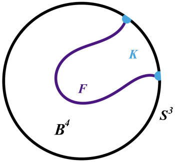

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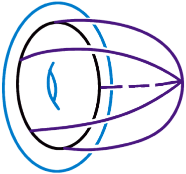

The 2-handle we attach along K with framing r

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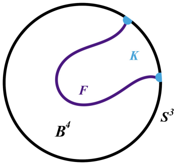

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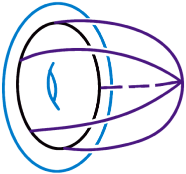

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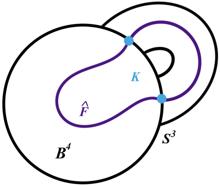

The core of the 2-handle caps the surface.

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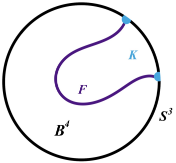

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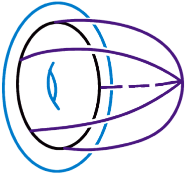

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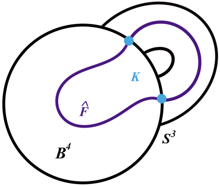

The core of the 2-handle caps the surface.

- There are knots that are not slice, but for some r there is a smoothly embedded S^2 that generates $H_2(X_r(K);\mathbb{Z}) \cong \mathbb{Z}$.

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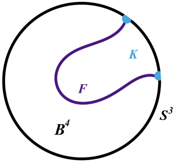

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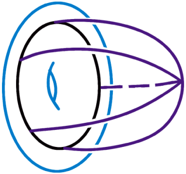

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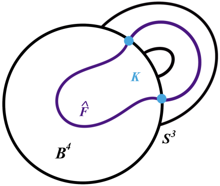

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- There are knots that are not slice, but for some r there is a smoothly embedded S^2 that generates $H_2(X_r(K);\mathbb{Z}) \cong \mathbb{Z}$.
- $g_{sh}^r(K)$ is called the *shake genus* of a knot K, and is defined to be the minimum genus of the surface generating $H_2(X_r(K);\mathbb{Z})$. We say K is *shake slice* when $g_{sh}^r(K) = 0$.

Knot Trace - Details

- The boundary of a knot trace is r-surgery,

$$\partial(X_r(K)) = S^3_r(K) := (S^3 \setminus \nu K) \bigcup D^2 \times S^1$$

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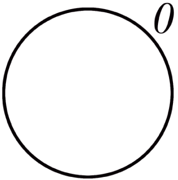
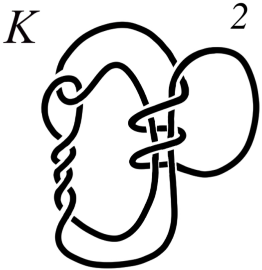
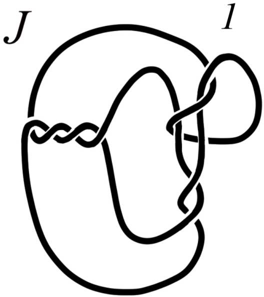
    - $S^2 \times D^2 = X_0(U)$



Examples

Akbulut (1976) showed that K is 2-shake slice and J is 1-shake slice using a sequence of blow ups and downs.





Non-Orientable Analog

Question

Do there exist knots K with $\gamma_4(K) > 1$, and some $r \in \mathbb{Z}$, so that a smoothly embedded $\mathbb{R}P^2$ generates $H_2(X_r(K);\mathbb{Z}_2) \cong \mathbb{Z}_2$?

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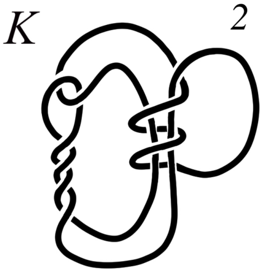
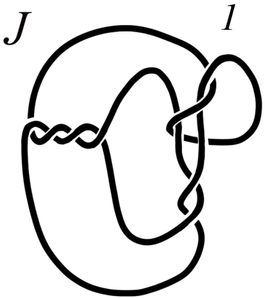
We define $\gamma^r_{sh}(K)$ to be the minimum genus of the non-orientable surface generating $H_2(X_r(K);\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Just use the existing examples?

Can we use the examples from Akbulut?





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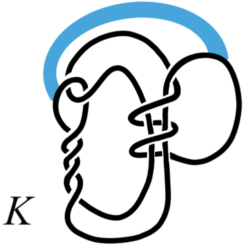

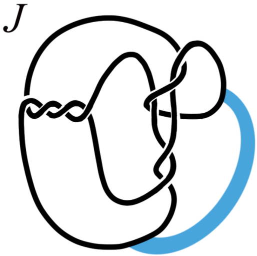

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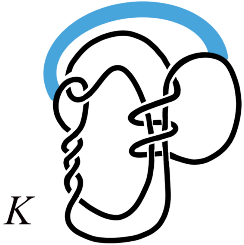

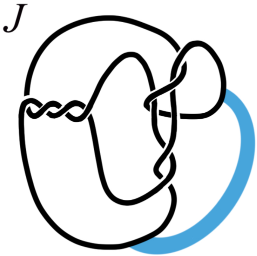

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Dissonance

Theorem (F)

For each genus g, there exists a $K \in S^3$ and $r \in \mathbb{Z}$ so that $g_{sh}^r(K) = g$ and $\gamma_{sh}^r(K) = 1$.

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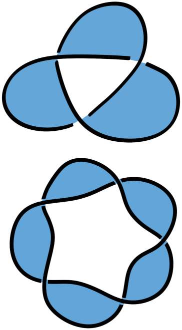

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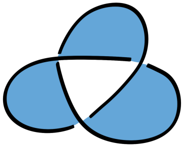

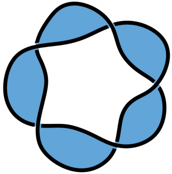

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3. For torus knots $T_{3,q}$, we have that for any relatively prime $q > 3$ and any $r < 2(q-1) - 1$, $g^r_{sh}(T_{3,q}) = g_4(T_{3,q}) = q - 1$ and $\gamma_4(K) = 1$.
   This covers cases for $g \geq 3$.

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Denote $\Upsilon_K(1)$ with a little upsilon $\upsilon(K)$.

Theorem (F)

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- [Piccirillo] Rasmussen's s-invariant is not a 0-trace invariant.

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Thank You!

Thank you for your attention!