

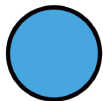
Non-Orientable Surfaces Bounded by Knots and the Knot Trace

Megan Fairchild

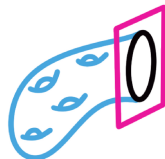
Louisiana State University

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Background - Slice Knots



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Definition (4-Genus)

Given a knot K in S^3 , the 4-genus, $g_4(K)$, is defined to be the minimum genus among all orientable surfaces S smoothly embedded in B^4 so that $\partial S = K$.

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When $g_4(K) = 0$, we say K is a *slice knot*.

Background - Non-Orientable Analog

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What are the obstructions for $\tau_4(K) = 1$?

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Thus, $_4(5_1) = 1$ and we have the bound $_4(K) \leq 2g_4(K) + 1$.

Techniques for Calculation

3 main methods for calculating the non-orientable 4-genus of knots.

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- ③ Obstructions from the double branched cover

Knot Invariants

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Lemma (F)

Given a knot K satisfying $\sigma(K) + 4\text{Arf}(K) \equiv 4 \pmod{8}$, and $c_4(K) \in \{1; 2\}$, then $\sigma_4(K) = 2$.

Knot Invariants - HFK

The little Upsilon invariant is denoted $\Upsilon(K)$.

Proposition (Ozváth–Stipsicz–Szabó)

Given K is a knot,

$$\Upsilon(K) - \frac{\tau(K)}{2} = \Upsilon_4(K)$$

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Theorem (Batson)

For a knot K ,

$$\frac{\chi(K)}{2} - d(S_{-1}^3(K)) = \Upsilon_4(K):$$

Band Moves & Cobordism

An orientable band move transforms a knot into a link.

Figure 8 knot to Hopf Link

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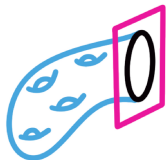
Figure 8 knot to Hopf Link

A non-orientable band move transforms a knot K into a different knot J .

Figure 8 knot to Trefoil

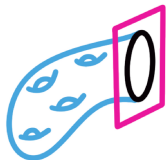
Transitioning to 4-Manifolds

Recall we have been discussing knots K in $S^3 = @B^4$ bounding surfaces in B^4 .



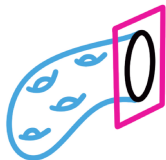
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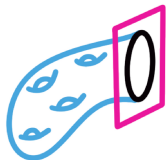
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Question (Minimal Genus Problem)

What is the minimal genus of an embedded surface which represents a two-dimensional homology class in a closed oriented smooth 4-manifold?

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and a F a surface in B^4
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There are knots that are not slice, but for some r there is a smoothly embedded S^2 that generates $H_2(X_r(K); \mathbb{Z}) \cong \mathbb{Z}$.

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$g_{sh}^r(K)$ is called the *shake genus* of a knot K , and is defined to be the minimum genus of the surface generating $H_2(X_r(K); \mathbb{Z})$. We say K is *shake slice* when $g_{sh}^r(K) = 0$.

Knot Trace - Details

The boundary of a knot trace is r -surgery,

$$\partial(X_r(K)) = S_r^3(K) = (S^3 \setminus K) \cup D^2 \times S^1$$

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$$S^2 \times D^2 = X_0(U)$$

Examples

Akbulut (1976) showed that \mathcal{K} is 2-shake slice and \mathcal{J} is 1-shake slice using a sequence of blow ups and downs.

Non-Orientable Analog

Question

Do there exist knots K with $\chi_4(K) > 1$, and some $r \in \mathbb{Z}$, so that a smoothly embedded $\mathbb{R}P^2$ generates $H_2(X_r(K); \mathbb{Z}_2) \cong \mathbb{Z}_2$?

Non-Orientable Analog

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Do there exist knots K with $\beta_4(K) > 1$, and some $r \in \mathbb{Z}$, so that a smoothly embedded $\mathbb{R}P^2$ generates $H_2(X_r(K); \mathbb{Z}_2) \cong \mathbb{Z}_2$?

We define $\beta_{sh}^r(K)$ to be the minimum genus of the non-orientable surface generating $H_2(X_r(K); \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Just use the existing examples?

Can we use the examples from Akbulut?

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$\tau_4(K) = 1 = \tau_4(J)$, per non-orientable band moves to slice knots.

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Dissonance

Theorem (F)

For each genus g , there exists a $K \subset S^3$ and $r \in \mathbb{Z}$ so that $g_{sh}^r(K) = g$ and $g_{sh}^r(K) = 1$.

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- 1 The Trefoil knot has $g_{sh}^0(3_1) = 1$ and $g_{sh}^0(3_1) = 1$.
- 2 The Cinquefoil knot has $g_{sh}^0(5_1) = 2$ and $g_{sh}^0(5_1) = 1$.

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- 1 The Trefoil knot has $g_{sh}^0(3_1) = 1$ and $g_4(K) = 1$.
- 2 The Cinquefoil knot has $g_{sh}^0(5_1) = 2$ and $g_4(K) = 1$.
- 3 For torus knots $T_{3;q}$, we have that for any relatively prime $q > 3$ and any $r < 2(q-1) - 1$, $g_{sh}^r(T_{3;q}) = g_4(T_{3;q}) = q - 1$ and $g_4(K) = 1$.

This covers cases for $g \geq 3$.

Invariants

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Theorem (F)

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[Hayden–Mark–Piccirillo] The concordance invariants Υ and Υ^* are not 0-trace invariants.

[Piccirillo] Rasmussen’s S -invariant is not a 0-trace invariant.

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Thank You!

Thank you for your attention!

