The h-Cobordism Theorem ~ Motivation and Background

[Smale 1960's, Fields medal]

*h-cobordism theorem: Let M^m and N^m be compact simply-connected oriented m-mflds that are h-cobordant through the simply-connected (m+1)-mfld W^{m+1}. if $m \geq 5$, then there is a diffeomorphism $W \cong M \times [0,1]$, which can be chosen to be the identity from $M \subset W$ to $M \times 0 \subset M \times [0,1]$. In particular, M and N must be diffeomorphic.

*Importance: Characterization of spheres.
the key in proving the generalized Poincaré conjecture in dim ≥ 5.
*Poincaré conjecture: if a smooth m-mfld Σ^m is homotopy equivalent to S^m, $m \geq 5$, then Σ^m & S^m are homeomorphic.
~Note, diffeomorphic fails in dim ≥ 7.

Recall

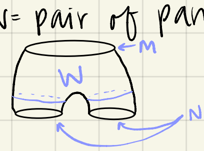
*defn: The n^{th} homotopy group, $\pi_n(X)$, is the group whose equivalence classes of maps $f: S^n \to X$ under (based) homotopy. That is, each map f must send some element $y \in S^n$ to x_0, and the homotopies F between the maps f must be based at x_0: $F_t(y) = x_0$ for all $0 \leq t \leq 1$.
- *A space X is connected if $\pi_0(X)$ is the trivial group.
- *A space X is simply-connected if $\pi_1(X)$ and $\pi_0(X)$ are both trivial.

*cobordism: A cobordism between two oriented m-mflds M and N is any oriented (m+1)-mfld W st its boundary is $\partial W = \bar{M} \cup N$. *Note: one of the mflds has reversed orientation!
~when such a W exists, M & N are called cobordant.

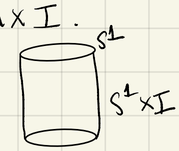
~examples:

$M = \{0\}$ dim = 0
$N = \{1\}$ dim = 0
$W = [0,1]$ dim = 1
~the unit interval as a cobordism between 2 points

$M = S^1$ dim = 1
$N = S^1 \sqcup S^1$ dim = 1
W = pair of pants dim = 2



! Trivial example !
M = M, an n-dim mfld.
$W = M \times I$.
eg:



cylinder gives trivial cobordism of S^1 to itself.

h-cobordisms are stronger than cobordisms

*A cobordism W between mflds M & N is an h-cobordism if it is homotopically like $M \times I$.

!Recall! A homeomorphism is a special case of a homotopy equivalence, in which $g \circ f = id_X$, $f \circ g = id_Y$. equal, not homotopic.

~equivalently~
- W deformation retracts to M (or N).
- the inclusion $M \hookrightarrow W$ is a homotopy equivalence. (or $N \hookrightarrow W$)
- if M & N are simply connected, this is equivalent to $H_*(W, M; \mathbb{Z}) = 0$.

*ex 1 & 3 from above are h-cobordisms.

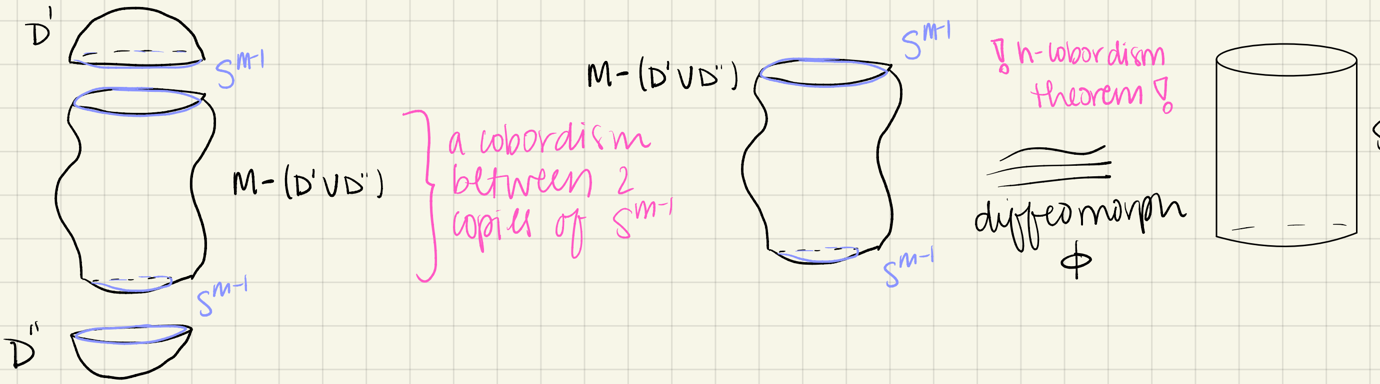
~broadly speaking: Given M & N two manifolds of dim ≥ 5, and W an h-cobordism between them. Then, M and N are diffeomorphic.

* proving the Poincaré conjecture in dimensions $n \geq 5$.

~ restated: For any $n \geq 6$, any simply connected, closed n-manifold M whose homology groups $H_p(M)$ are isomorphic to $H_p(S^n)$ for all $p \in \mathbb{Z}$ is homeomorphic to S^n.

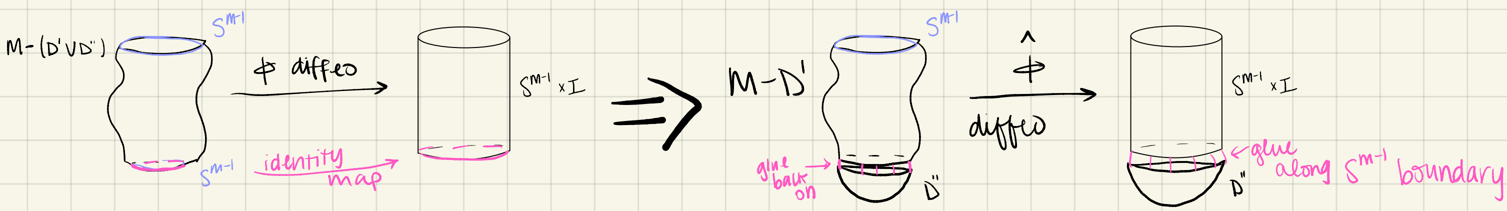
*note: dim = 5, 6 the statement can be strengthened for a diffeomorphism, $M \overset{\sim}{=} S^n$.

*pf: let M be a manifold of $\dim = m \geq 6$.
cut out 2 small m-dim disks, D' and D''.



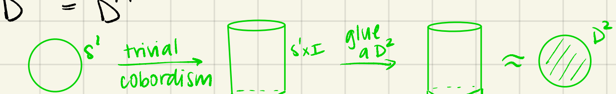
* Now we wish to extend this map Φ to include D' and D''.
(caution: Φ may not extend to a diffeomorphism)

* first, we notice that the diffeomorphism $\Phi : M-(D' \cup D'') \longrightarrow S^{m-1} \times I$ is the identity map on the bottom S^{m-1}. According to the schematic below, we now have a diffeomorphism
$\hat{\Phi} : M - D' \longrightarrow (S^{m-1} \times I) \cup D''$

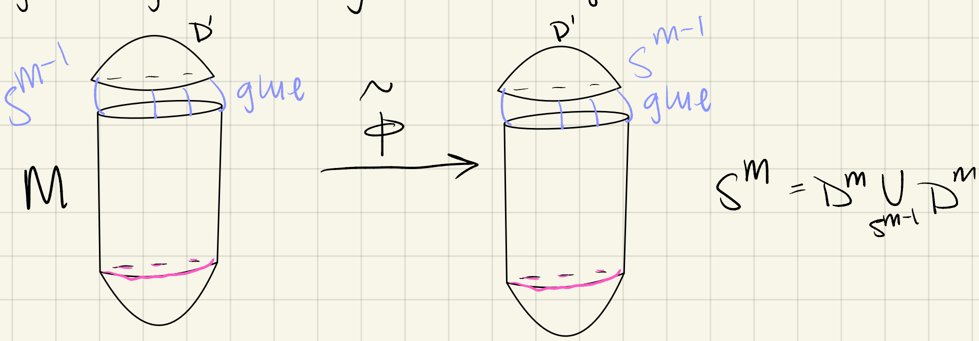


observe $S^{m-1} \times I \cup D'' = D^m$

↳ try to see it:



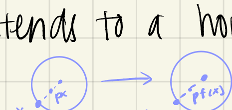
* We may now view M as being constructed from two m-dim disks, D^m and D', that are glued together along S^{m-1}. Now glue the D' to both $M-D'$ and D^m $(= S^{m-1} \times I \times D'')$



$S^m = D^m \cup_{S^{m-1}} D^m$

* Note $\hat{\Phi}$ extends to a diffeomorphism over the S^{m-1}.

* Note: Any diffeomorphism of boundary spheres S^{m-1} extends to a homeomorphism of D'.
→ the extension is a radial extension.



* this completes the proof for $\dim \geq 6$. ∎