* Introduction to handles & handlebodies *

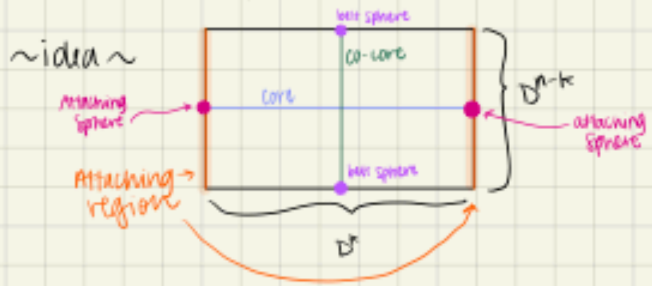
I What is a handle?

*defn: For $0 \le k \le n$, an n-dimensional k-handle h is a copy of $D^k \times D^{n-k}$.

~anatomy of a handle~

1. Attaching region: $\partial D^k \times D^{n-k} = S^{k-1} \times D^{n-k}$
2. Attaching sphere: denoted $A^k = \partial D^k \times \{0\} = S^{k-1} \times \{0\}$
3. core: denoted $C^k = D^k \times \{0\}$
4. co-core: denoted $K^k = \{0\} \times D^{n-k}$
5. belt sphere: denoted $B^k : \{0\} \times \partial D^{n-k} = \{0\} \times S^{n-k-1}$

~idea~



n-dim k-handle

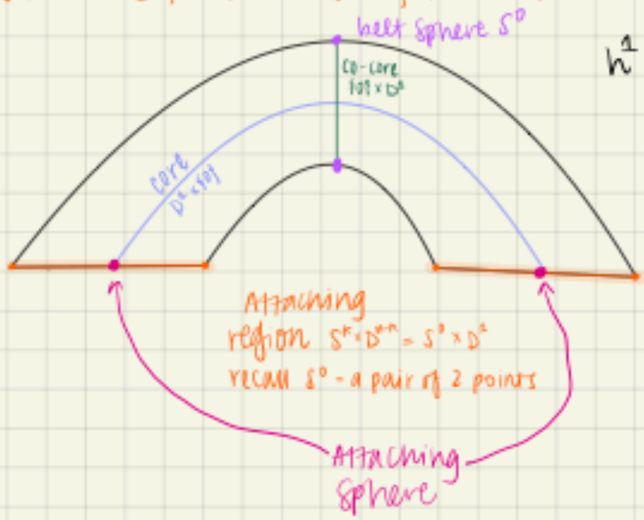
~concrete example: 2D-manifolds

2-dim 0-handle $h_0 : D^0 \times D^2 = D^2$

2-dim 1-handle $h_1 : D^1 \times D^1$

2-dim 2-handle $h_2 : D^2 \times D^0 = D^2$

~let's draw h_1, the 1-handle, $D^1 \times D^1$. D^1 is an "edge", —•, so $D^1 \times D^1$ = a band

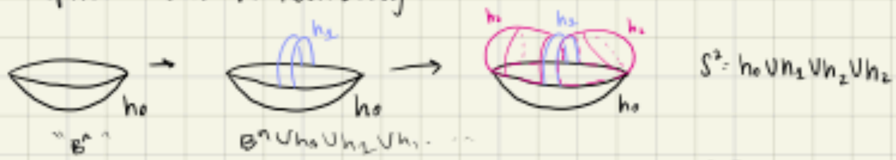


*Homework: write out the 3-dimensional handles h_0, h_1, h_2, and h_3. Can you draw h_1 and h_2 & label the 5 regions? (h_2 is more difficult).

Ⅱ constructing manifolds using handles

* A handlebody is a compact manifold expressed as a union of handles.

ex: A 2-sphere as a handlebody



* How do we use handles to construct manifolds ~ abstractly?

we start with a manifold X and "glue" the handles to the boundary of X, denoted ∂X.

If we start with the 0-handle for 3-manifolds, we start with B^3, the 3-dim'l ball.

... "glue" seems mysterious, can we be more precise?

* Techical details *

Denote $\varphi: \partial D^k \times D^{n-k} \to \partial X$ as the attaching map.

3-manifold specifics: $\varphi: \underbrace{\partial D^k \times D^{3-k}}_{\text{this is the attaching region!}} \to \partial X$ $\quad$ φ is continuous, differentiable

So our maps are $\varphi_1: S^0 \times D^2 \to \partial B^3 = S^2$. we specify the "feet" of the 1-handle by stating where disks (D^2) are mapped on the boundary of a 3-ball, which is S^2.

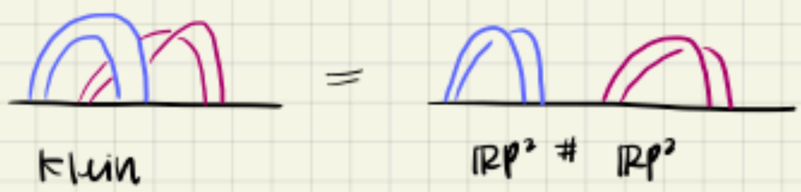
For φ_2, mapping the 2-handles, we do not simply have B^3 anymore, we now have $B^3 \cup h_1$, or B^3 union all the 1-handles we have attached. This is why the map is "∂X" as once we start adding handles, we begin to change the space.

* Homework: compare your handle drawings to this idea. Does it make sense?

Think about the ideas of a "handlebody" and how each handle is attached for both 2 & 3 dimensional manifolds.

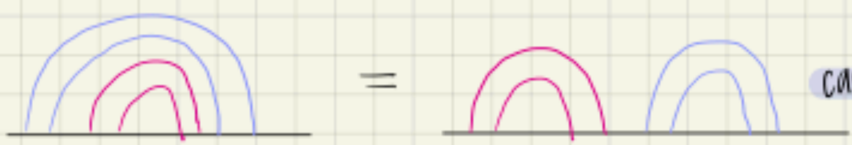
handle slides & drawing 3-mflds

~ The homework was to show:



by using handle slides.

*additionally consider:



can you explain why?

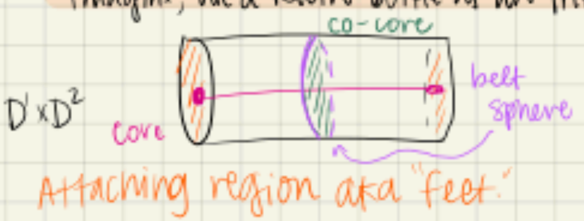
Handle decompositions for 3-manifolds

* 0-handles: $h_0 = D^0 \times D^3$, $A_0 = \emptyset$ "poofs into existence"

* 1-handles: $h_1 = D^1 \times D^2$, $A_1 = \partial D^1 \times D^2 = S^0 \times D^2$

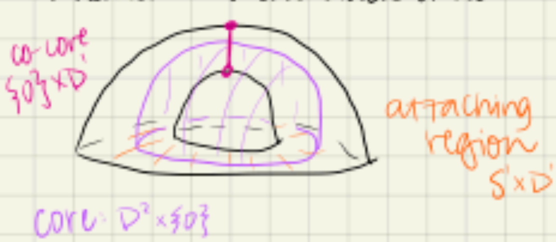
*Attaching 1-handles we have to worry about orientation.

HW: Think about what a non-orientable handle attachment might "look like" (very hard to imagine, use a Klein bottle as an inspiration.) ← will discuss next time.



* 2-handles: $h_2 = D^2 \times D^1$, $A_2 = S^1 \times D^1$ aka igloos.

~ we do not worry about orientation here. Attaching h_2 to a mfld X does not affect whether X is orientable or not.

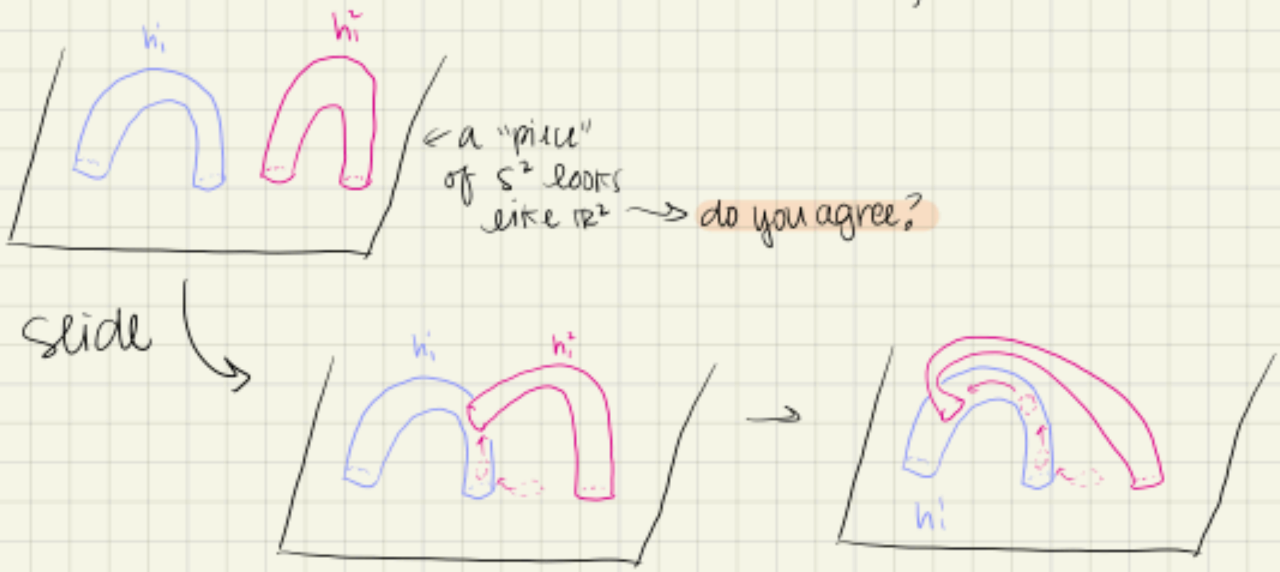


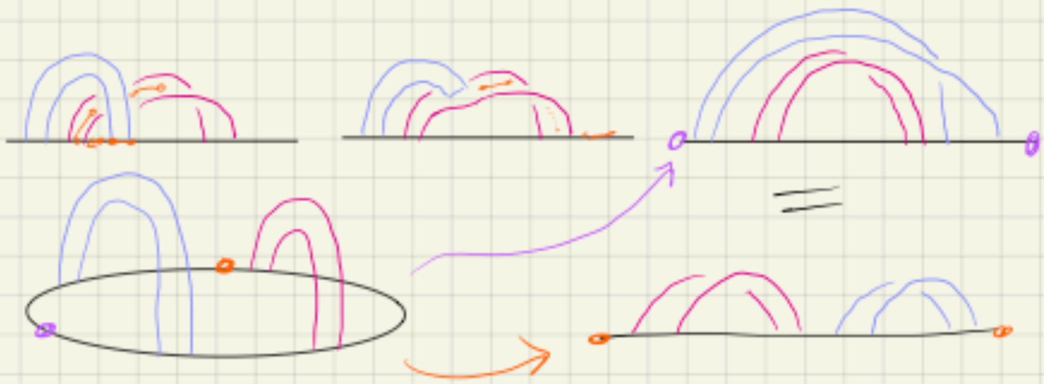
* 3-handles: $h_3 = D^3 \times D^0$, $h_3 = S^2 \times D^0$ "caps" or "hats" the manifold.

FACT:

* Given X is a closed, oriented 3-mfld, we may assume the handle decomposition consists of only one 0-handle and one 3-handle. When we attach 1-handles, the attaching regions "live" in $\partial h_0 = \partial B^3 = S^2$.

~ slide some one handles ~   notation: h_1^1 = 1-handle #1, h_1^2 = 1-handle #2.



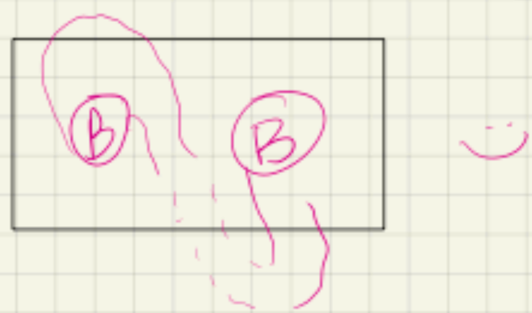


* Drawing 3-mflds

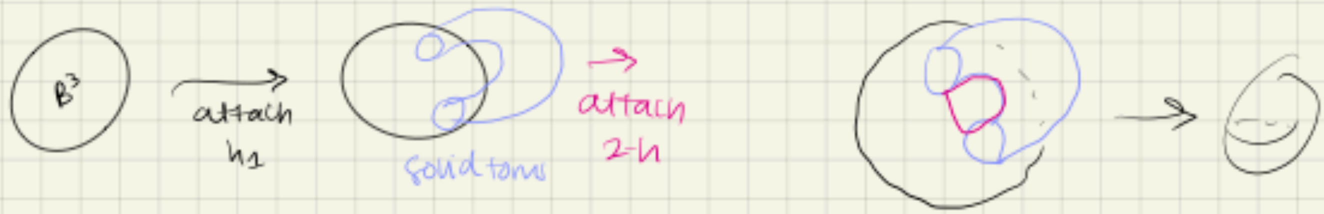
① start $h_0 = B^3$ ☺
② attach handles to $\partial B^3 = S^2$ ☺
③ Locally, $S^2 = \mathbb{R}^2$ ☺
④ therefore, I draw handle attachments on a "sheet of paper" ☺
⑤ I can draw oney the attaching regions.

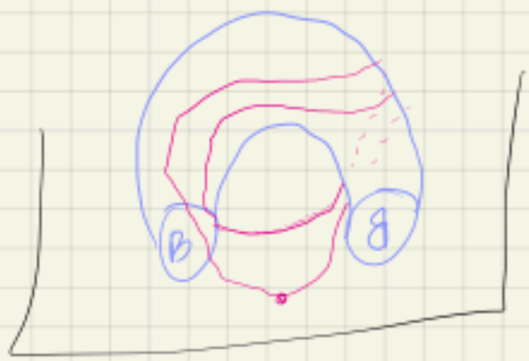
$\partial B^3 = S^2$

non-orientable:



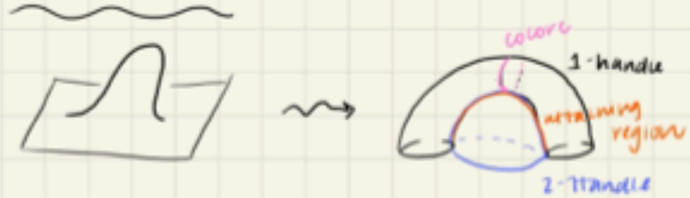
* Drawing the attaching maps of the 2-handles.
  - ~ attach h_2 along $\partial X = \partial(B^3 \cup h_1^k)$ k h_1's
  - ~ attaching region for $h_2 = D^2 \times D^1$ is $S^1 \times D^1$ CIRCLES
* handles can "cancel" called "cancelling pair"





* canceling pairs
* what is hopf fibration
  ↳ 3-sphere & tori
* What is a lens space

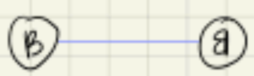
*cancelling pairs



Proposition 4.2.9. *A $(k-1)$-handle h_{k-1} and a k-handle h_k $(1 \le k \le n)$ can be cancelled, provided that the attaching sphere of h_k intersects the belt sphere of h_{k-1} transversely in a single point.*

*Restate for 3-mflds: if h_2, a 2-handle, and h_1, a 1-handle, are attached to X so that the attaching sphere of h_2 intersects the belt sphere of h_1 exactly one time, then $X \cup h_2 \cup h_1 \cong X$

*ex: Heegaard diagram for S^3



this is the standard decomposition of S^3 as the union of 2 solid tori.

~ Standard decomposition? So how do we see S^3 as 2 solid tori? How does this Heegaard diagram make sense??

*We start with the Hopf fibration (there is a wiki page, if needed).

① What is a fibration? A triple:

"FIBER" $F \xrightarrow{\text{embedding}} E$ "TOTAL SPACE"

$\downarrow$ π or p is conventional name of this map.

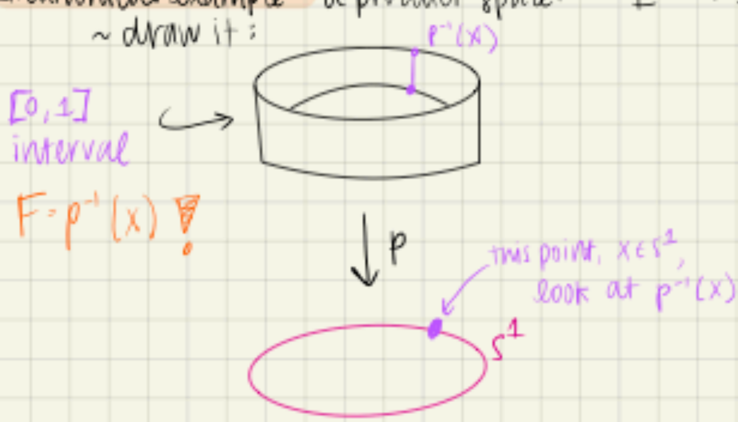
B "BASE SPACE"

the preimage of any point in B is F.

*canonical example: a product space.

$$\underset{\text{interval}}{I} \hookrightarrow S^1 \times I = \text{annulus}$$
$$\downarrow p$$
$$S^1$$

~ draw it:



*pre-image of the point is the fiber!

Hopf fibration

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^3 \\ & & \downarrow p \\ & & S^2 \end{array}$$

For each $x \in S^2$, the fiber is S^1.
important distinction: $S^2 \times S^1 \neq S^3$.
the total space E ($= S^3$ in this moment) is often ~~not~~ <u>not</u> $F \times B$, in this case $S^1 \times S^2$.

coordinates: in $\mathbb{R}^4$. Denote angle ξ_1, ξ_2 (greek letter Xi) as angle that can take any value between 0 and 2π. Angle η (greek letter Eta) where $0 \leq \eta \leq \pi/2$.

~ in $\mathbb{R}^4$:
$$x_0 = \cos\xi_1 \sin\eta$$
$$x_1 = \sin\xi_1 \sin\eta$$
$$x_2 = \cos\xi_2 \cos\eta$$
$$x_3 = \sin\xi_2 \cos\eta$$

HW For any fixed value of η, show (ξ_1, ξ_2) parameterize a 2D-torus.
hint: don't overthink. Use calculus 3 and drawings. Don't be too formal. just convince yourself this works.

~ what happens when $\eta = 0$ or $\pi/2$?

HW How does this relate to the Heegaard diagram?

$h_0: B^3$
$h_1: B^1 \times B^2$
$h_2: B^2 \times B^1$
$h_3: B^3$

hint: what is a 0-handle ∪ 1-handle?
what is a 2-handle ∪ 3-handle?

~ this is a challenge question we haven't discussed yet. Just think on it, no pressure.

* Construction of 3-spheres and fibrations *

~ let us first look at a geometric approach to the construction of spheres. (unit spheres here)

recall: S^n = n-dimensional sphere, B^n = n-dimensional ball (sometimes D^n for disk).

① construct S^1 from two B^1:

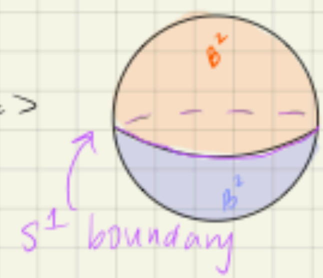
B^1 •—• •—• B^1 glue along boundary (S^0)

S^0 boundary

$= S^1$ defined by $x^2+y^2=1$.

② construct S^2 from two B^2:

S^1 boundary S^1 boundary } Glue along boundary =>



$= S^2$

$x^2+y^2+z^2=1$.

③ construct S^3 from 2 B^3.

glue along S^2 boundary.

$S^3 = \{x, y, z, w \mid x^2+y^2+z^2+w^2=1\}$ in $\mathbb{R}^4$.

~ this is one common construction of higher dimensional spheres.
(can't draw it though)

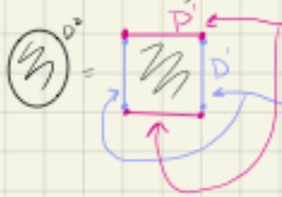
*The 2 tori:

S^3 is the boundary of $B^4 = D^2 \times D^2$, and the boundary of $D^2\times D^2 = S^1\times D^2 \cup D^2 \times S^1$

=> $S^3 = S^1\times D^2 \cup D^2\times S^1$ (this is cheeky and perhaps non-helpful)

however, let's apply this same idea to S^1, which is the boundary of $D^2 = D^1\times D^1$.

③ D^2 =



now view $S^1 = \boxed{S^0\times D^1} \cup \boxed{D^1\times S^0}$ = boundary of $D^1\times D^1$.

~ now let's do maps & fibrations.

* first, let's ground ourselves in our idea of a "preimage"

← preimage of this piece of the parabola is an open interval.

- recall that f is continuous if for every open set in the codomain, its preimage is open in the domain.
$f: X \to Y$, $U \subseteq Y$ open $\Rightarrow f^{-1}(U) \subseteq X$ open.

* Now we want a map $p: \underset{\text{total space}}{E} \to \underset{\text{base space}}{B}$ so that at every point $b \in B$, there is a neighborhood U of b so that $p^{-1}(U) \subseteq E$ is homeomorphic to $U \times \underset{\text{fiber}}{F}$ via $h: p^{-1}(U) \to U \times F$ so that "h commutes with projection onto U."

The maps:

$$\begin{tikzcd} F \arrow[r, "\text{inclusion}"] & E \arrow[dl, "h"'] \arrow[d, "p"] \\ U \times F \arrow[r, "\text{projection onto } U"'] & B \end{tikzcd}$$

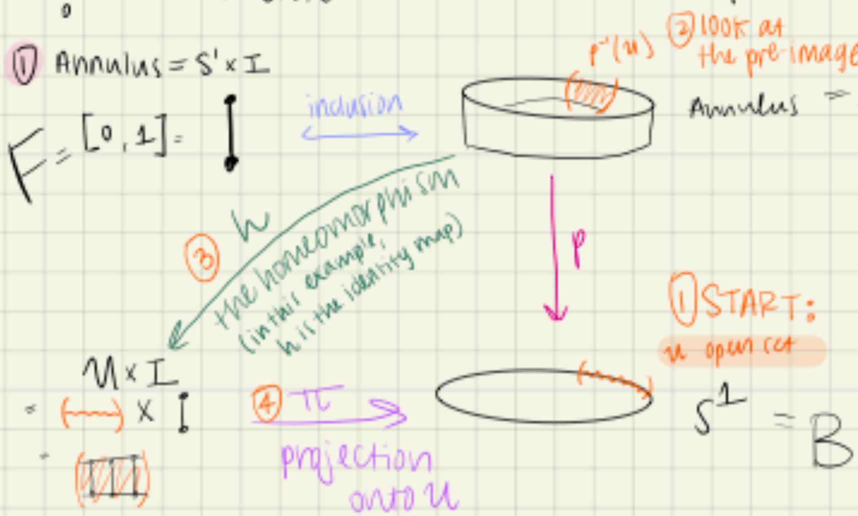
$p^{-1}(U)$ b U

this is the same "U"

recall projection map: $(x,y) \in U \times F$. $\pi: U \times F \to B$ gets rid of the F component, $\pi(x,y) = x \in B$. This works because $U \subseteq B$.

Make it concrete. Start with trivial examples.

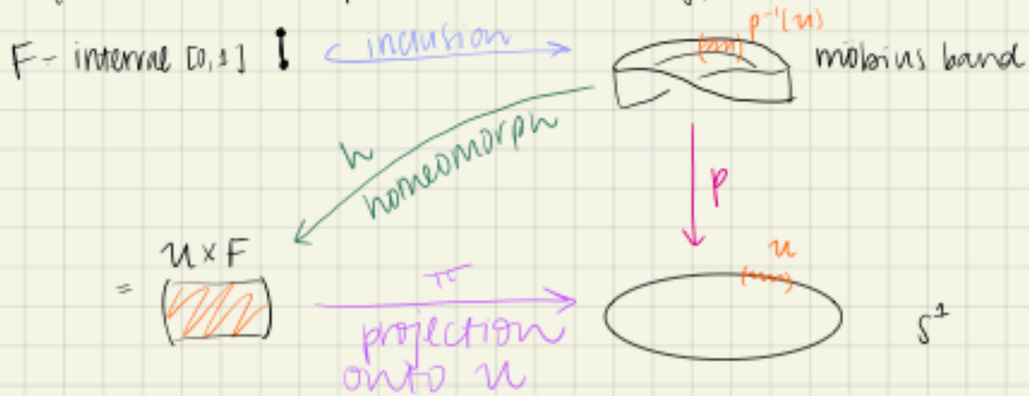
① Annulus $= S^1 \times I$



*HW: can you see how $\pi \circ h = p$?
This is the main idea of a fibration, also called "fiber bundle."

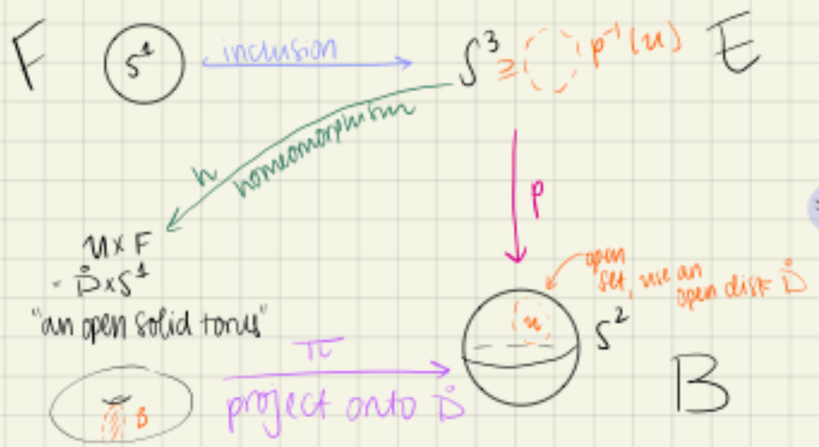
Trivial fiber bundles are when $E = B \times F$. Here, annulus $= S^1 \times I \Rightarrow$ trivial.

② A non-trivial example: the möbius band as a fiber bundle.
(note: when we end up with non-orientable things, we call them <u>twisted</u> bundles (twisted fiber bundles)).



* Here, the möbius band $\neq S^1 \times I$ = annulus.
Fibers are defined locally by neighborhoods. Locally, the neighborhoods look like $S^1 \times I$, but globally our E-total space is not $S^1 \times I$. The twisted bundles are denoted $S^1 \tilde{\times} I$.
↳ twisted cross product

③ The Hopf fibration - a non-trivial fiber bundle.



*Another way to think: "above every point in S^2 is an S^1" this means given $(x,y,z) \in S^2$, $p^{-1}((x,y,z)) = S^1$.

*hw: can you see why using $\pi \circ h = p$?

LENS SPACES

~ we do the group theoretic approach first ~

*Lens spaces are 3-manifolds denoted $L(p,q)$ where $\gcd(p,q)=1$ and $0 \le q < p$ by convention. We do a brief review of terminology before defining a lens space.

*defn: A group G **acts** on a set X if there is a map $F: G \times X \to X$ where, if we denote $f(g,x)$ by $g \cdot x$, then $1 \cdot x = x \ \forall x \in X$, 1 is identity in G, & $g(h \cdot x) = (gh) \cdot x \ \forall g,h \in G$, $x \in X$

~ if X is a topological space, the mapping $\theta_g : X \to X$ given by $\theta_g(x) = g \cdot x$ is a homeomorphism and X is called a G-space.

~ if X is a G-space st $g \cdot x = x$ for some x implies $g = 1$, then G acts freely on X

*Given G acts on X, X/G denotes the set of equivalence classes $\{[x] \mid x \in X\}$ where $[x_1] = [x_2]$ iff $x_1 = gx_2$ for some $g \in G$. X/G is called the **orbit space** of X over G, and the $G \cdot x$ constitute a collection of disjoint orbits, corresponding to the equivalence classes.

recall $\mathbb{C}^2 \sim \mathbb{R}^4$ and $S^3 \subset \mathbb{R}^4$

*defn: Define the 3-sphere S^3 to be the set $\{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$

*recall the cyclic group with p elements is $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ (p not necessarily prime) and addition is defined (mod p) where 0 is the additive identity.

~ Defining a lens space ~

*Fix some $q \in \mathbb{Z}$ with $0 \le q < p$ and $\gcd(p,q) = 1$, and let $\mathbb{Z}_p$ act on S^3 as follows:

$m \in \mathbb{Z}_p$, $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$

$\Phi_m : S^3 \to S^3$ given by $\Phi_m(z_0, z_1) = \left(e^{\frac{2\pi i m}{p}} \cdot z_0,\ e^{\frac{2\pi i q m}{p}} \cdot z_1\right)$

~ this is a group action ~

① $0 \in \mathbb{Z}_p$: $0(z_0, z_1) = (e^0 z_0, e^0 \cdot z_1) = (z_0, z_1)$ ✓

② $m, n \in \mathbb{Z}_p$: $m(n \cdot (z_0, z_1)) = m(e^{\frac{2\pi i n}{p}} z_0, e^{\frac{2\pi i n q}{p}} \cdot z_1) = \left(e^{\frac{2\pi i (m+n)}{p}} z_0, e^{\frac{2\pi i (m+n) q}{p}} \cdot z_1\right) = (m+n)(z_0, z_1)$.

~ facts ~

Φ_m is bijective, continuous, has a continuous inverse $\Rightarrow$ homeomorphism.

$\Rightarrow S^3$ is a $\mathbb{Z}_p$-space. Also, $\mathbb{Z}_p$ acts freely on S^3.

We now define $L(p,q)$ to be the orbit space $S^3/\mathbb{Z}_p$ with respect to the action.

*Fact: the fundamental group of all the lens spaces $L(p,q)$ is $\mathbb{Z}_p$ (regardless of q).

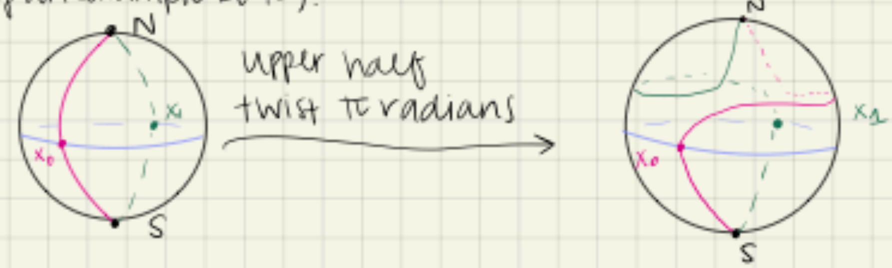
~geometric approaches~

① solid 3-ball. Choose p for $L(p,q)$.

these are evenly spaced.
$\{x_0, \dots, x_{p-1}\}$ represent elements in $\mathbb{Z}_p$.

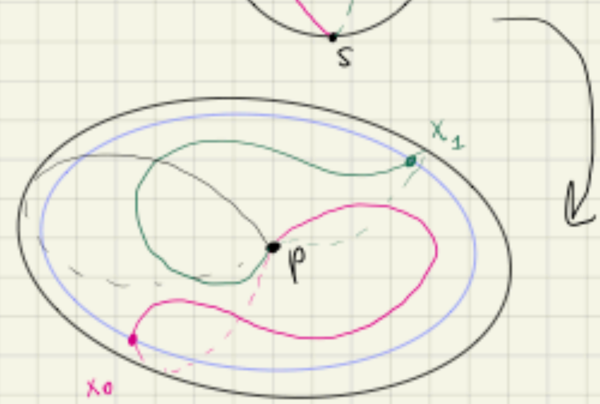
we then identify the upper-half with the lower half after a $\frac{2\pi q}{p}$-radian positive rotation of the upper half wrt the lower.

~try an example $L(2,1)$.



*we then identify the north & south pole

It becomes difficult to draw, but the next step is to identify x_0 & x_1. sometimes this step is called "identify spherical triangles."



p is the point of the north & south identification

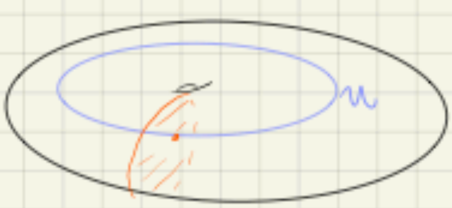
② Dehn surgery on knots

$L(p,q)$ can be defined by p/q surgery along the unknot (eg, S^1).

Some notation: u = the unknot. $\nu(u)$ = a neighborhood of the unknot.

notes on $\nu(u)$: the neighborhood of a point is a disk
thus, the neighborhood of a knot (any knot) is a solid torus

meridian = μ longitude = λ



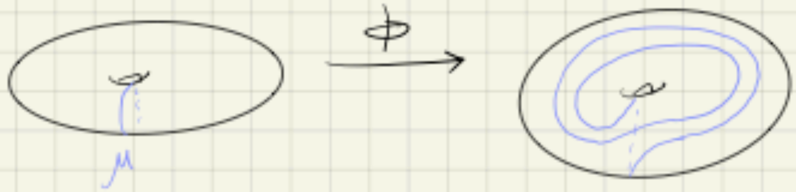
(common notation & language for the torus in general)

construct $L(p,q)$

① Remove $\nu(u)$, the neighborhood of the unknot, which is a solid torus.
$S^3 - \nu(u)$

② "Glue" or "sew" back in another solid torus via some homeomorphism ϕ.
both terms are used for this operation.
ST = solid torus.
$\phi: \partial(ST) \to \partial(ST)$ and $\partial(ST) = T^2$, boundary of solid torus is a torus.
so we have $\phi: T^2 \to T^2$ defined by $\phi(\mu) = q\lambda + p\mu$. We send the meridian to twist q times longitudinally & p times around the meridian.

$L(2,1)$:



HW How would we draw this as a handle diagram?

*Draw L(2,1):

instead of drawing arcs in B³, we assume they are there.

want to draw only the feet and blue arcs

L(2,1):

*L(3,1):

*L(3,2)