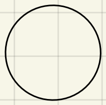
* A Quick Intro to Knot Concordance

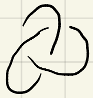
~ Start at the start ~

A knot K is an embedding $S^1 \hookrightarrow S^3$.

~ examples of knots ~

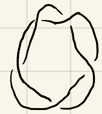
m = mirror

0_1  the unknot (trivial knot)

3_1  right-handed trefoil

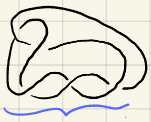
$m3_1$  left-handed trefoil

4_1  figure-8

5_1  cinquefoil

6_1  stevedore

we will often refer to knots that are not named by their name in the Rolfsen knot table, which is (crossing number) — index. For example, the 5_2 knot is drawn below.

5_2 knot: 

note the difference from the 6_1 (stevedore) knot.

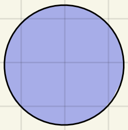
* determining when 2 knots are "the same" is quite a challenge, but we can use surfaces to build a type of equivalence between knots, known as concordance.

* All knots bound surfaces, but such surfaces come in different flavors:
  1. orientable surfaces, Seifert surfaces, live in 3-dimensional space.
  2. surfaces that live in the 4-ball, B^4.
  3. Two knots can co-bound a surface (knot concordance & cobordism).

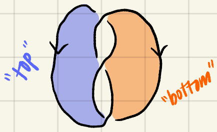
① Seifert surface (always orientable!)

~ there is an algorithm (Seifert's Algorithm) that can be used to construct such surfaces. We won't go over the algorithm in these notes.

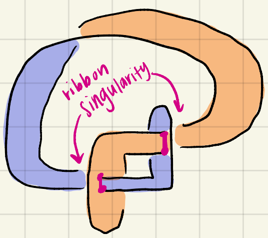
~ examples:



the unknot bounds a disk



the trefoil bounding a genus 1 orientable surface

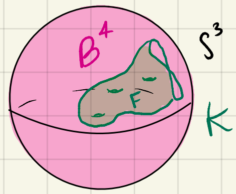


this is an example of a knot bounding a disk with ribbon singularities. Knots that bound disks with ribbon singularities are called ribbon knots.

② Knots bounding surfaces in B^4

~ the set up:

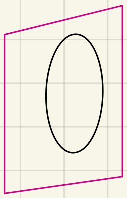
We know S^3 bounds B^4, so K in S^3 bounds a surface F in B^4.



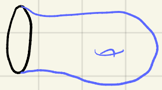
This is hard to imagine!

~ A 3D Analog ~

* consider the unknot, drawn in 2-dimensional space.



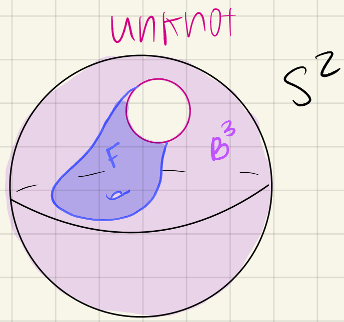
the unknot is the boundary of a torus (with boundary)



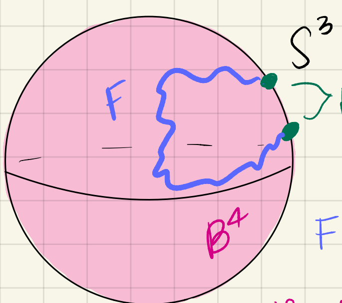
this cannot be drawn in 2D-space!

so we have the unknot in S^2 bounding $\underbrace{\text{a torus (with bdry)}}$ in B^3:

call this F, the surface bounded by the knot.



~ we try to draw a similar picture for knots in S^3 and surfaces in B^4, but as we cannot draw 4D pictures, we lower everything by 1 dimension.



the 3-sphere drawn as a 2D sphere

K the knot is $S^1 \hookrightarrow S^3$ drawn as $S^0 \hookrightarrow S^2$, 0-dimensional boundary of a 1-dimensional object.

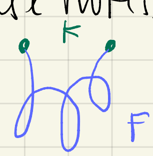
F a 2D Disk drawn as a line (1D) with endpoints the knot.

the 4-ball drawn as a 3D ball.

~ these schematics are to help us visualize and not to be 100% accurate.

~ notes on F:

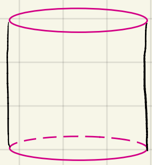
all of our embeddings (for these notes) are smooth. Thus, we do not have disks like:



* A knot is called slice if it bounds a disk D^2 in B^4. eg, the unknot is slice. the Stevedore (6_1) knot is also slice.

③ Two knots co-bounding a surface.

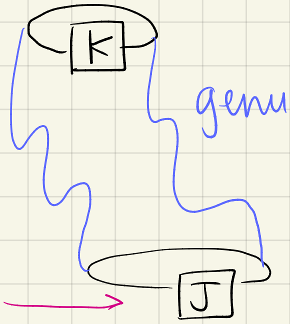
~ simple example:



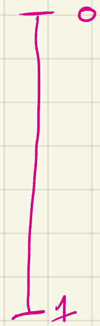
* two unknots co-bound a genus zero surface

* Two knots K and J are **concordant** if there is a smooth embedding of a genus zero surface into $S^3 \times [0,1]$ where K is in $S^3 \times \{0\}$ and J is in $S^3 \times \{1\}$.

$S^3 \times \{0\}$



$S^3 \times \{1\}$



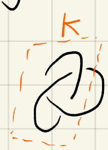
the interval

* the surface can have saddles (and likely will).

* the genus 0 surface is often called "an embedded annulus."

~ a note on the drawing:

when we draw arbitrary knots, we do  or  a box with an arc. why?

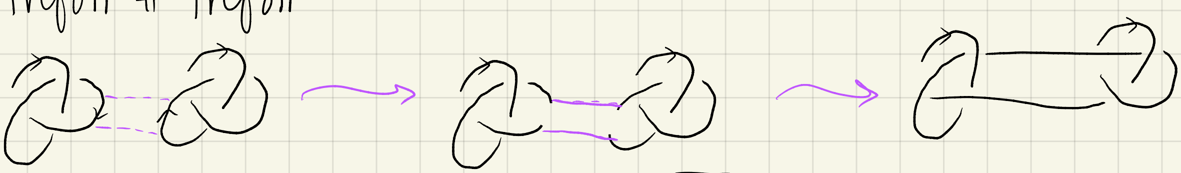
take a simple example: , so 

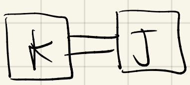
in the box is the knot.

* The knots K and J are concordant if K # -J is slice.

~ connect sum of knots.

trefoil # trefoil

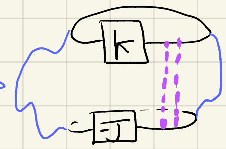


~ drawn abstractly: 

CAN you PROVE IT?

HINT

Annulus →



~ what is -J? The mirror reverse

trefoil = K   reverse = K^r mirror = mK   mirror reverse = $-K$ (mK^r)

* knot concordance is an equivalence relation. try to prove it!

① show reflexive: $K \sim K$ hint: $K \# -K$

② show symmetric: $K \sim J \Rightarrow J \sim K$. hint: draw a picture.

③ show transitive: $K \sim J$, $J \sim M \Rightarrow K \sim M$.

* We are following "A survey of classical knot concordance" - Chuck Livingston.

*** Thm 2.2:** The set of concordance classes of knots forms a countable abelian group, denoted $\mathcal{C}$, with operation connect sum and the unknot representing the identity.

~ can you show $\mathcal{C}$ is a group? inverse?

$\boxed{\text{RECALL:}}$ Slice knots bound a smoothly embedded disk in B^4.

~ slice knots are concordant to the unknot ~

*** Defn 2.3:** A knot K is called **ribbon** if it bounds an embedded disk D in B^4 for which the radial function on the ball restricts to a smooth Morse function with $\underbrace{\text{no local maxima}}_{\text{eg, no index 2 critical points.}}$ in the interior of D.

Slice-ribbon conjecture: A knot is slice $\Leftrightarrow$ A knot is ribbon. → still open!

§2.2 - Algebraic Concordance

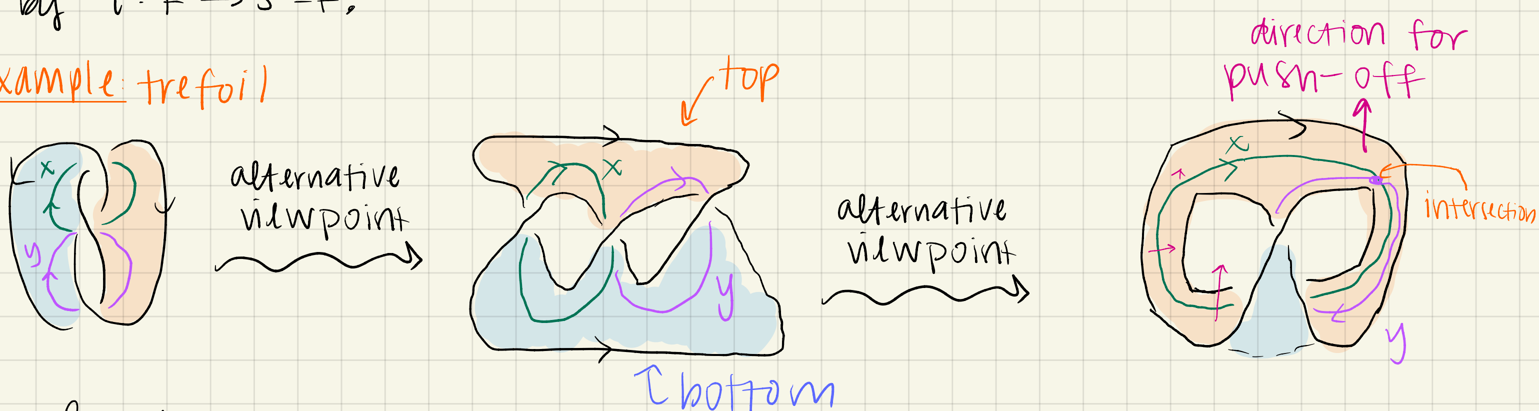
*** Defn 2.4:** K is a knot with Seifert surface F. A **Seifert pairing** is a bilinear map

$$\boxed{V : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}}$$

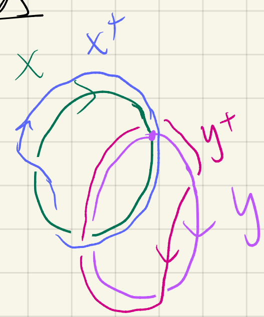
where $V(x, y) = lk(x, i_* y)$ sometimes written $lk(x, y^+)$

$i_* y$ or y^+ denotes the positive push off. i_* is the map induced by $i : F \to S^3 - F$.

example: trefoil



loops:



$lk(x, x^+) = -1$
$lk(x, y^+) = 1$

$lk(y, y^+) = -1$
$lk(y, x^+) = 0$

Seifert matrix: $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

note: