

Non-Orientable Surfaces in Knot Traces

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What about the non-orientable surface generating $H_2(X; \mathbb{Z}_2)$?

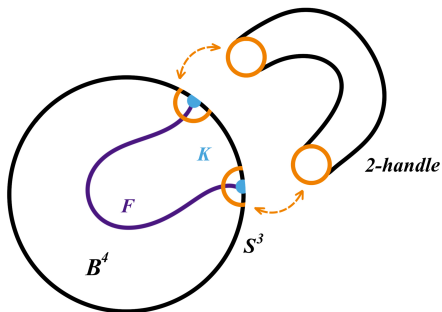
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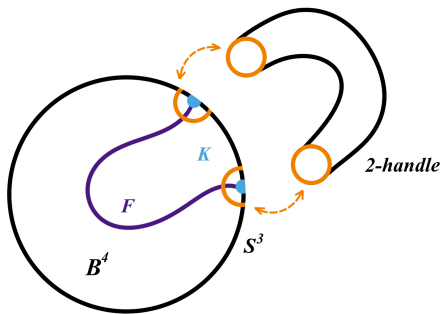
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$X_r(K)$ is called the r -trace of K .

Background

Definition (Shake Genus)

The shake-genus of K , $g_{sh}^r(K)$, is the genus of the surface generating $H_2(X_r(K); \mathbb{Z}) \cong \mathbb{Z}$. A knot is called r -shake slice if $g_{sh}^r(K) = 0$ for some $r \in \mathbb{Z}$.

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Lemma (F)

For any knot K , $g_{sh}^r(K) \geq 2g_{sh}^r(K) + 1$.

Preliminary Results

Question

Given a knot K in S^3 such that K does not bound a Möbius band, does there exist an $r \in \mathbb{Z}$ so that a smoothly embedded $\mathbb{R}P^2$ generates $H_2(X_r(K); \mathbb{Z}_2)$?

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Proposition (F)

The obstructions for a knot being r -shake slice do not hold in the non-orientable setting.

Examples

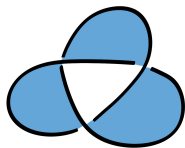
Theorem (F)

For each genus g , there exists a $K \subset S^3$ and $r \in \mathbb{Z}$ so that the genus of the orientable surface $S = H_2(X_r(K); \mathbb{Z})$ is g and the genus of the non-orientable surface $F = H_2(X_r(K); \mathbb{Z}_2)$ is 1.

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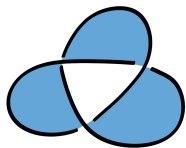


- 1 The Trefoil knot has $g_{Sh}^r(K) = g_4(K) = 1$ for every r and $g_4(K) = 1$.
- 2 The Cinquefoil knot has $g_{Sh}^r(K) = g_4(K) = 2$ for every r and $g_4(K) = 1$.

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- 3 For torus knots $T_{3;q}$, we have that for any relatively prime $q > 3$ and any $r < 2(q-1) - 1$, $g_{Sh}^r(T_{3;q}) = g_4(T_{3;q}) = q - 1$ and $g_4(K) = 1$.

This covers cases for $g \geq 3$.

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Theorem (F)

Suppose P is a winding number one pattern with \tilde{P} slice. Given a knot K is related to a knot J by one non-orientable band move, then $P(K)$ is related to $P(J)$ by one non-orientable band move.

Satellite Results

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Given the Mazur pattern P , $\chi_4(P_r(K)) = \chi_4(K) + 1$ for any knot K and all $r \in \mathbb{Z}$.

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Corollary (F)

Given the pattern P is the Mazur pattern, $\chi_4(\tilde{P}_r) = 1$ for all $r \in \mathbb{Z}$.

Thank You

Question

Are there knots where $r_{sh}^r(K) > 1$ for every r ?