Non-Orientable Surfaces in Knot Traces

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- Knots provide a method of constructing 4-manifolds so that the surface generating the second homology has arbitrarily high genus.
- What about the non-orientable surface generating $H_2(X;\mathbb{Z}_2)$?

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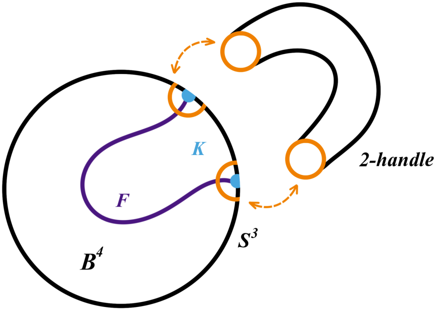

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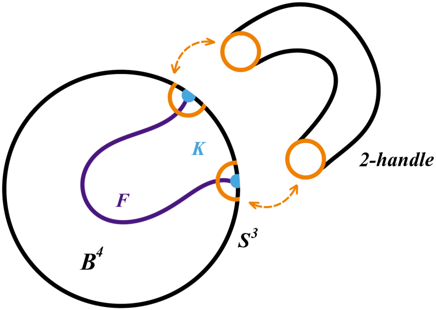
- $X_r(K)$ is called the *r-trace of K*.



Background

Definition (Shake Genus)

The shake-genus of K, $g_{sh}^{r}(K)$, is the genus of the surface generating $H_2(X_r(K);\mathbb{Z}) \cong \mathbb{Z}$. A knot is called r-shake slice if $g_{sh}^{r}(K) = 0$ for some $r \in \mathbb{Z}$.

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Lemma (F)

For any knot K, $\gamma^r_{sh}(K) \leq 2g^r_{sh}(K) + 1$.

Preliminary Results

Question

Given a knot K in S^3 such that K does not bound a Möbius band, does there exist an $r \in \mathbb{Z}$ so that a smoothly embedded $\mathbb{R}P^2$ generates $H_2(X_r(K); \mathbb{Z}_2)$?

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There exists 4-manifolds X so that the genus of the orientable surface S generating $H_2(X;\mathbb{Z}) \cong \mathbb{Z}$ is strictly greater than the genus of the non-orientable surface F generating $H_2(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

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Proposition (F)

The obstructions for a knot being r-shake slice do not hold in the non-orientable setting.

Examples

Theorem (F)

For each genus g, there exists a $K \in S^3$ and $r \in \mathbb{Z}$ so that the genus of the orientable surface $S \in H_2(X_r(K);\mathbb{Z})$ is g and the genus of the non-orientable surface $F \in H_2(X_r(K);\mathbb{Z}_2)$ is 1.

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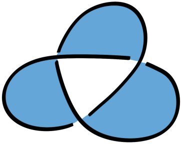

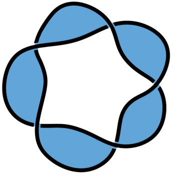

1. The Trefoil knot has $g^r_{sh}(K) = g_4(K) = 1$ for every r and $\gamma_4(K) = 1$.
2. The Cinquefoil knot has $g^r_{sh}(K) = g_4(K) = 2$ for every r and $\gamma_4(K) = 1$.

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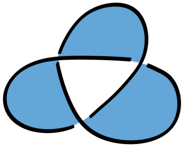

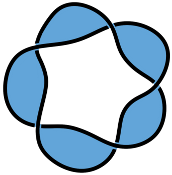

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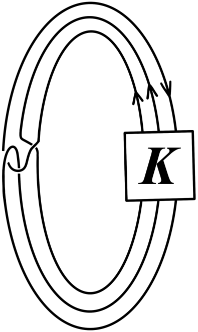
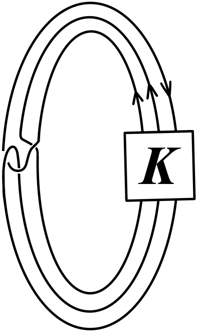
3. For torus knots $T_{3,q}$, we have that for any relatively prime $q > 3$ and any $r < 2(q-1) - 1$, $g^r_{sh}(T_{3,q}) = g_4(T_{3,q}) = q - 1$ and $\gamma_4(K) = 1$.
   This covers cases for $g \geq 3$.

Satellite Knots



- Consider a winding number one pattern P with $\tilde{P} = P(U)$ a ribbon knot.



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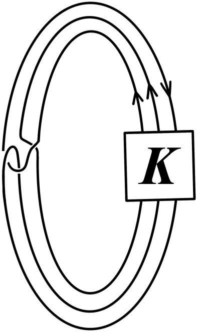

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Theorem (F)

Suppose P is a winding number one pattern with $\tilde{P}$ slice. Given a knot K is related to a knot J by one non-orientable band move, then $P(K)$ is related to $P(J)$ by one non-orientable band move.

Satellite Results

Proposition (F)

Given the the Mazur pattern P, $\gamma_4(P_r(K)) \leq \gamma_4(K) + 1$ for any knot K and all $r \in \mathbb{Z}$.

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Corollary (F)

Given the pattern P is the Mazur pattern, $\gamma_4(\tilde{P}_r) = 1$ for all $r \in \mathbb{Z}$.

Thank You

Question

Are there knots where $\gamma^r_{sh}(K) > 1$ *for every* r*?*