Non-Orientable 4 Genus of Knots

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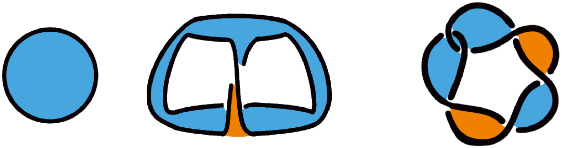
Background - Slice Knots



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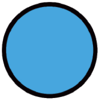


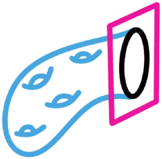

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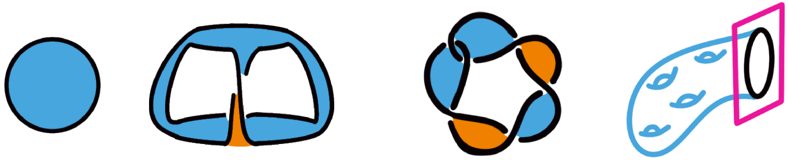

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Given a knot K in S^3, the 4-genus, $g_4(K)$, is defined to be the minimum genus among all orientable surfaces S smoothly embedded in B^4 so that $\partial S = K$.

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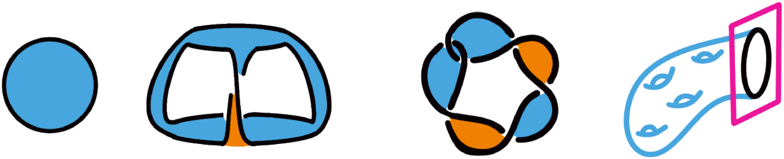

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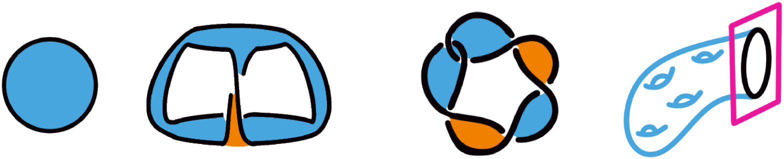

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- When $g_4(K) = 0$, we say K is a *slice knot*.

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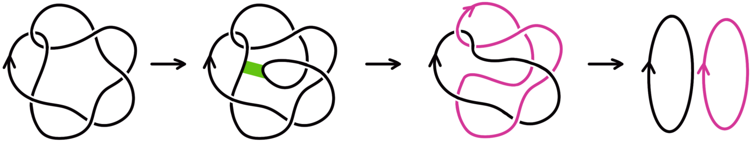
The orientable band move from the 6_1 knot to 2 unlinked unknots.



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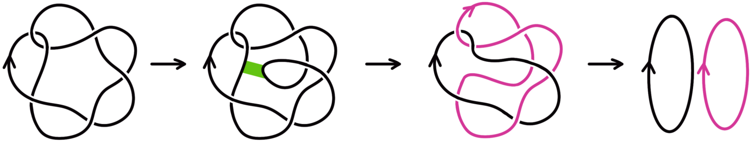
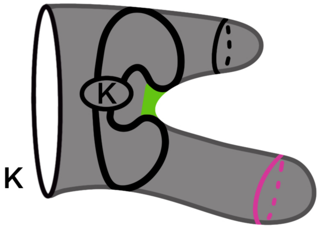


A view of the knot bounding a disk in B^4.



Background - Non-Orientable Analog

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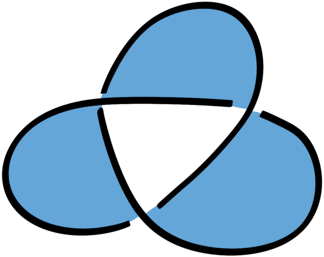

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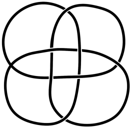

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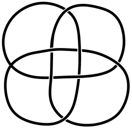

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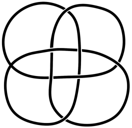

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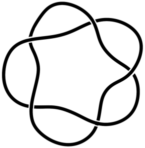

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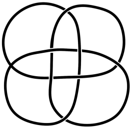

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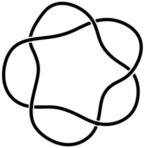

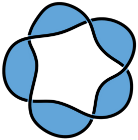

Thus, $\gamma_4(5_1) = 1$ and we have the bound $\gamma_4(K) \leq 2g_4(K) + 1$.

Main Results

Theorem (F)

For the 185 non-alternating 11 crossing knots,

- (a) *121 knots have* $\gamma_4(K) = 1$
- (b) *58 knots have* $\gamma_4(K) = 2$

The remaining 6 knots have $\gamma_4(K) = 1$ *or* 2.

Techniques for Calculation

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1. Lower bounds coming from knot invariants
2. Non-orientable band moves
3. Obstructions from the double branched cover

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Example: The figure 8 knot has $\sigma(4_1) = 0$ and Arf $(4_1) = 1$ and thus $\gamma_4(4_1) \geq 2$.

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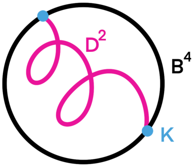

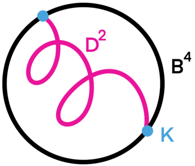

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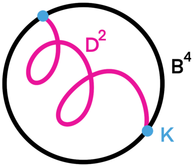

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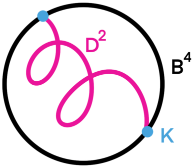
Proposition (Murakami–Yasuhara)

For any knot K,

$$\gamma_4(K) \leq \begin{cases} c_4(K) & \textit{if } c_4(K) \textit{ is even and } c_4(K) \neq 2 \\ c_4(K) + 1 & \textit{otherwise} \end{cases}$$



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Question: Is there any way to improve this for a precise result?

Knot Invariants

Lemma (F)

Given a knot K satisfying $\sigma(K) + 4\mathrm{Arf(K)} \equiv 4 \pmod 8$, and $c_4(K) \in \{1, 2\}$, then $\gamma_4(K) = 2$.

Knot Invariants - HFK

The little Upsilon invariant is denoted $\upsilon(K)$.

Proposition (Ozváth–Stipsicz–Szabó)

Given K is a knot,

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There is a recursive formula for torus knot calculations.

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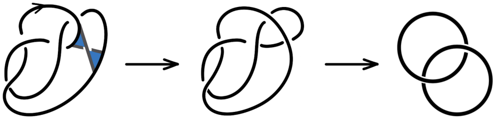
Theorem (F-Garcia-Murphy-Percle)

For a torus knot $T_{p,q}$ where $p < q$,

$$d(S^3_{-1}(T_{p,q})) = 2\left(\left\lfloor \frac{p}{2} \right\rfloor + \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor - 1} \left\lfloor \frac{(p-1-2k)q - p - 1}{2p} \right\rfloor\right).$$

Orientable Band Moves

An orientable band move transforms a knot into a link.



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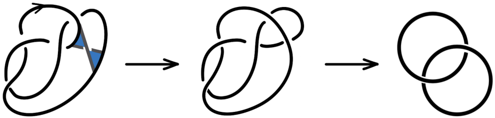

Figure 8 knot to Hopf Link

A non-orientable band move transforms a knot K into a different knot J.

Figure 8 knot to Trefoil

Non-Orientable Band Moves

Proposition (Jabuka–Kelly)

If the knots K and K' are related by a non-oriented band move, then

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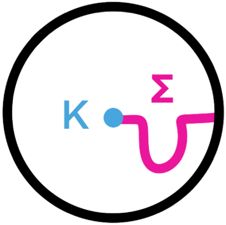
Example: The figure 8 knot, 4_1, has $\gamma_4(4_1) \geq 2$. 4_1 is related to the trefoil by one non-oriented band move. Thus, $\gamma_4(4_1) = 2$.

Double Branched Cover

Denote the double branched cover of S^3 over a knot K as $D_K(S^3)$.



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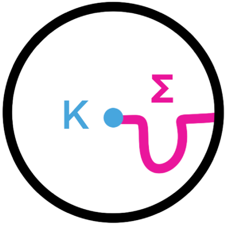

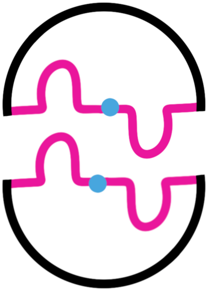

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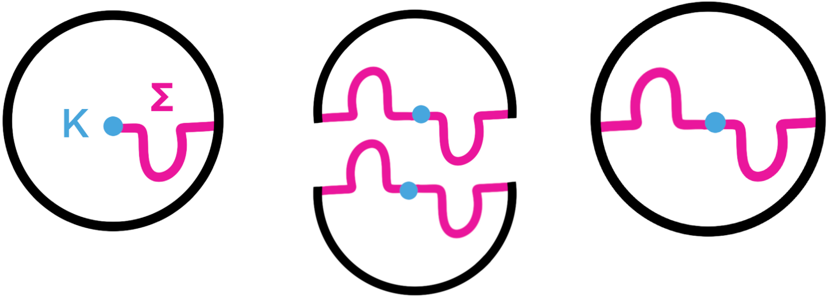

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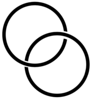

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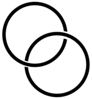

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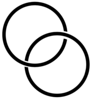

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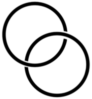

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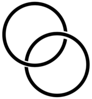

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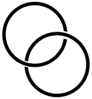

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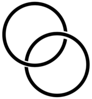

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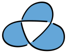
The linking form can be directly calculated from a checkerboard coloring of a knot. 

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Corollary (Gilmer–Livingston)

Suppose that $H_1(D_K(S^3)) = \mathbb{Z}_n$ *where* n *is the product of primes, all with odd exponent. Then if* K *bounds a Möbius band in* B^4*, there is a generator* $a \in H_1(D_K(S^3))$ *such that* $\lambda(a,a) = \pm 1/n$

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Corollary (F)

Let K in S^3 be a knot and suppose that $H_1(D_K(S^3)) = \mathbb{Z}_{p^2q}$ where p is prime and q is a product of primes, all with odd exponent. Then if K bounds a Möbius band in B^4, there is a generator $a \in H_1(D_K(S^3))$ such that either $\lambda(a,a) = \pm 1/p^2q$ or $\lambda(a,a) = \pm 1/q$.

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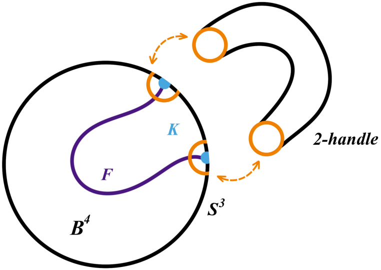
Thus, the knots $11n_{17}$, $11n_{40}$, $11n_{159}$, $11n_{166}$, $11n_{177}$ and $11n_{178}$ all have $\gamma_4(K) = 1$ or 2.

Future Work



- Let K be a knot in S^3 and a F a non-orientable surface in B^4 where $\partial F = K$.
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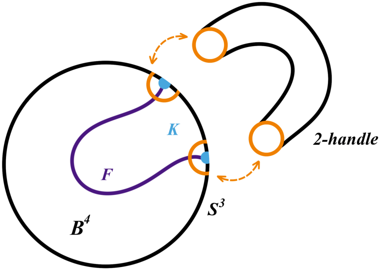

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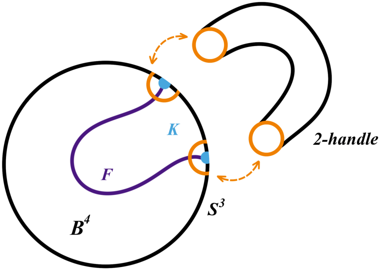

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Question

Do there exist knots K with $\gamma_4(K) > 1$, and some $r \in \mathbb{Z}$, so that a smoothly embedded $\mathbb{R}P^2$ generated $H_2(X_r(K);\mathbb{Z}_2) \cong \mathbb{Z}_2$?

Examples

Theorem (F)

For each genus g, there exists a $K \in S^3$ and $r \in \mathbb{Z}$ so that the genus of the orientable surface $S \in H_2(X_r(K);\mathbb{Z})$ is g and the genus of the non-orientable surface $F \in H_2(X_r(K);\mathbb{Z}_2)$ is 1.

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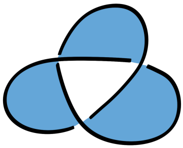

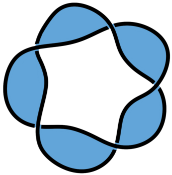

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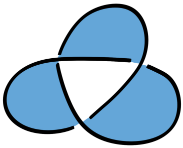

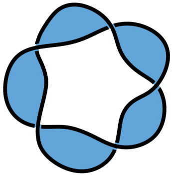

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2. The Cinquefoil knot has $= g_4(K) = 2$ for every r and $\gamma_4(K) = 1$.
3. For torus knots $T_{3,q}$, we have that for any relatively prime $q > 3$ and any $r < 2(q-1)-1$, $g^r_{sh}(T_{3,q}) = g_4(T_{3,q}) = q-1$ and $\gamma_4(K) = 1$.
   This covers cases for $g \geq 3$.

Thank You!

Thank you for your attention!